Grade 7/8 Math Circles
March 25th - 28th, 2024
Continued Fractions

Section 1: Types of Numbers

In mathematics we work with a ton of different numbers. We work with whole numbers (integers), positive and negative numbers, even and odd numbers, and irrational and rational numbers. You might not know what irrational and rational numbers are, but today we are going to look at these two types of numbers more closely and see what makes them so different to one another! Most importantly we are going to learn how to express these vastly different numbers using the idea of **continued fractions**!!

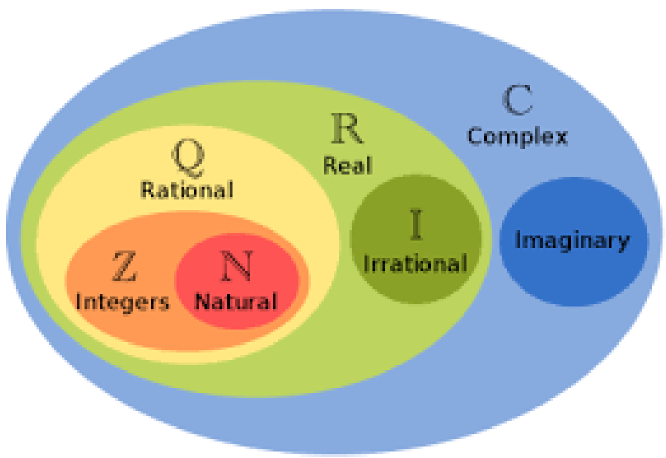

Figure 1: Types of Numbers

Section 2: Rational Numbers

Definition 1

A rational number is a fraction that can be written in the form $\frac{a}{b}$ where both a and b are integers and $b \neq 0$.

Notice not all fractions are rational numbers. Based off of Definition 1, for example 3.5/2.5 is a fraction which can be rewritten into the form $\frac{a}{b}$ where both a and b are integers.

Stop and Think

Is there a way we could write the fraction $\frac{3.5}{2.5}$ in the form $\frac{a}{b}$ where both a and b are integers?

Example 2.1

Write the fraction $\frac{3.5}{2.5}$ in the form $\frac{a}{b}$ where both a and b are integers?

Solution:

Recall that a rational number is a fraction that can be written in the form $\frac{a}{b}$ where both a and b are integers and $b \neq 0$.

We know that $3.5 = 7/2$ and similarly we know that $2.5 = 5/2$ so the fraction $3.5/2.5$ can be rewritten as $\frac{7/2}{5/2}$.

Now dividing by a fraction is equivalent to multiplying by the reciprocal of that fraction. So $\frac{7/2}{5/2} = \frac{7}{2} \times \frac{2}{5} = \frac{14}{10}$. Therefore, $\frac{3.5}{2.5}$ can be written in the form $\frac{a}{b}$ where both a and b are integers, given by $\frac{14}{10}$

Exercise 2.1

Write the fraction $\frac{5.5}{2.5}$ into the form $\frac{a}{b}$ where both a and b are integers.

Exercise 2.1 Solution

Recall that a rational number is a fraction that can be written in the form $\frac{a}{b}$ where both a and b are whole numbers and $b \neq 0$.

We know that $5.5 = 11/2$ and similarly we know that $2.5 = 5/2$ so the fraction $5.5/2.5$ can be rewritten as $\frac{11/2}{5/2}$.

Now dividing by a fraction is equivalent to multiplying by the reciprocal of that fraction. So

$$\frac{11/2}{5/2} = \frac{11}{2} \times \frac{2}{5} = \frac{22}{10} = \frac{11}{5}.$$

Let's begin classifying some numbers and determining if they are rational numbers or fractions based off of Definition 1 and both Example 2.1 and Exercise 2.1.

Exercise 2.2

Which of the following are rational numbers and which are not? If the number is a rational number then determine if you can rewrite it in the form $\frac{a}{b}$ where both a and b are integers.

1. $\frac{1}{2}$
2. $\frac{10745}{347}$
3. 2
4. $\frac{1/2}{3/4}$
5. $\frac{-3.5}{5.7}$
6. $\frac{\pi}{2}$

Exercise 2.2 Solution

1. Yes, $\frac{1}{2}$ fulfills Definition 1
2. Yes, $\frac{10745}{347}$ fulfills Definition 1
3. Yes, $\frac{2}{1}$ fulfills Definition 1
4. $\frac{1/2}{3/4}$ is a fraction. But we know that $\frac{1/2}{3/4} = \frac{1}{2} \times \frac{4}{3} = \frac{4}{6}$, which fulfills Definition 1.
5. $\frac{-3.5}{5.7}$ is a fraction. But we know that $-3.5 = \frac{-7}{2}$ and $5.7 = \frac{57}{10}$. So $\frac{-3.5}{5.7} = \frac{-7/2}{57/10} = \frac{-7}{2} \times \frac{10}{57} = \frac{-70}{114}$ which fulfills Definition 1.
6. $\frac{\pi}{2}$ is a fraction but since there is no way to rewrite π as a rational number we cannot turn $\frac{\pi}{2}$ into a rational number.

Stop and Think

What types of fractions are not rational numbers?

Section 3: Finite Continued Fractions

So far we've defined rational numbers as fractions that can be written in the form $\frac{a}{b}$ where both a and b are whole numbers and $b \neq 0$. For the simplicity of the lesson we will focus on working with the continued fraction expansions of positive rational numbers. Let's begin exploring **finite** continued fractions!

Definition 2

A finite continued fraction has the form:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

where $a_1, \cdots, a_n$ are positive whole numbers and a_0 is a positive whole number or 0. We can denote a finite continued fraction in two ways, the first being as seen above and the second $[a_0, a_1, \cdots, a_n]$.

Right now this seems very odd and you may be questioning why we would use this. Let's first look at some examples to see how this may be helpful!

Example 3.1

Identify the rational number which has the continued fraction expansion [1, 2, 3, 2].

Solution:

Let's begin by writing out the given expansion as a fraction so we can better visualize what number this could be, writing it out we get; $1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{2}}}$

Now the rest of this question comes down to manipulating fractions starting from the denominator.

1. $3 + \frac{1}{2} = \frac{7}{2}$, so we get $1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{2}}} = 1 + \cfrac{1}{2 + \cfrac{1}{\frac{7}{2}}}$.

2. $\frac{1}{\frac{7}{2}} = \frac{2}{7}$, so we get $1 + \cfrac{1}{2 + \cfrac{1}{\frac{7}{2}}} = 1 + \frac{1}{2 + \frac{2}{7}}$.

3. $2 + \frac{2}{7} = \frac{16}{7}$, so we get $1 + \frac{1}{2 + \frac{2}{7}} = 1 + \frac{1}{\frac{16}{7}}$.

4. $\frac{1}{\frac{16}{7}} = \frac{7}{16}$, so we get $1 + \frac{1}{\frac{16}{7}} = 1 + \frac{7}{16}$.

5. Finally $1 + \frac{7}{16} = \frac{23}{16}$

Therefore the rational number which has the continued fraction expansion [1, 2, 3, 2] is $\frac{23}{16}$.

Activity 1

Now it's your turn to practice!! Get into pairs or groups of three or four and have one person come up and pick a continued fraction expansion. Your task as a group is to identify the rational number which it represents. Once you're done come and check your answer with me :)!

Example 3.2

Solve for the continued fraction expansion of $\frac{89}{37}$.

Solution:

If we perform long division on 89 and 37 we will see that 37 goes into 89 twice and leaves a remainder of 15, so we can begin our rewrite as follows: $\frac{89}{37} = 2 + \frac{15}{37}$. Now basing our steps off of the backwards operations we did in Example 3.1 we can rewrite $\frac{15}{37} = \frac{1}{\frac{37}{15}}$. Now that we have performed these two steps it is time we repeat them for the fraction $\frac{37}{15}$.

If we perform long division on 37 and 15 we will see that 15 goes into 37 twice and leaves a remainder of 7, so we can begin our rewrite as follows: $\frac{37}{15} = 2 + \frac{7}{15}$. Which we rewrite as $2 + \frac{7}{15} = 2 + \frac{1}{\frac{15}{7}}$. Putting this together with out first iteration we have a continued fraction that looks like $2 + \frac{1}{2 + \frac{15}{7}}$.

If we perform long division on 15 and 7 we will see that 7 goes into 15 twice and leaves a remainder of 1, so we can begin our rewrite as follows: $\frac{15}{7} = 2 + \frac{1}{7}$. Which we rewrite as $2 + \frac{1}{7} = 2 + \frac{1}{\frac{7}{1}} = \frac{1}{7}$. Putting this together with out first and second iterations we have a continued fraction that looks like $2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{7}}}$. We know we have reached our end point here because $\frac{1}{7}$ is already expressed as a finite continued fraction $\frac{1}{a}$. Therefore the continued fraction expansion of $\frac{89}{37}$ is [2, 2, 2, 7].

Activity 2

Now it's your turn to check each other's work!! Trade the rational numbers you solved for in Activity 1 with a group near you and solve for the continued fraction expansion of that number. Once you are done check your answer with the group you traded with :)!

Section 4: Irrational Numbers

So far we've seen how friendly rational numbers are and how simple we can represent their continued fraction expansions. Unfortunately, not all numbers are as nice to work with as the rationals. We are now going to learn about irrational numbers and it will be through working with irrational numbers that we will see why continued fraction expansions are so helpful!

Definition 3

An irrational number is a number which is not rational, i.e it cannot be expressed as a fraction of integers.

Irrational numbers have two defining qualities;

1. Irrational numbers have decimal expansions which are non-terminating (go on infinitely).
2. Irrational numbers have decimal expansions which are non-recurring (they don't follow a pattern at any point).

Exercise 4.1

Which of the following are irrational numbers and which of the following are rational numbers?

1. $\frac{1}{2}$
2. $2.\bar{6} = \frac{8}{3}$
3. $\sqrt{2}$
4. $\frac{1/\pi}{1/\pi}$
5. $\frac{-3.5}{5.7}$
6. $\frac{\pi}{2}$

Exercise 4.1 Solution

1. No, $\frac{1}{2}$ has a terminating decimal, so it is rational.
2. No, $2.\bar{6}$ is recurring, so it is rational.
3. Yes, $\sqrt{2}$ has a non-terminating and non-recurring decimal.
4. No, $\frac{1/\pi}{1/\pi} = 1$ which is rational.
5. No, $\frac{-3.5}{5.7} = \frac{-35}{57}$ which is a fraction of integers.
6. Yes, $\frac{\pi}{2}$ has a non-terminating and non-recurring decimal.

Section 5: Infinite Continued Fractions

Here's where things get interesting, we've seen how vastly different irrational numbers are from rational numbers, so we'd expect to see something interesting happen with the continued fraction expansions of irrational numbers.

Definition 4

An infinite continued fraction has the form:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

where $a_1, \cdots, a_n$ are positive whole numbers and a_0 is a positive whole number or 0.

If our continued fraction was finite we could represent it in two ways; by writing out the fraction expansion as a whole or by writing out the terms $[a_0, a_1, \cdots, a_n]$, but how do we represent infinite continued fractions?

Definition 5

If an infinite continued fraction exhibits a recurring pattern, for example it is of the from $[a, b, a, b, a \cdots]$, then we denote this pattern by placing a bar over the repeating elements of the expansion. That is $[a, b, a, b, a \cdots] = [\bar{a, b}]$.

Helpful Tools

As everything with irrational numbers we'll see that they are a bit harder to work with so we'll need two tools to be able to build and solve infinite continued fraction expansions.

Our first tool is actually something that you already saw during last week's lesson, it is the quadratic formula!

Definition 6

The quadratic formula is a tool to solve for x in equations of the form $ax^2 + bx + c$, the quadratic formula takes on the following form;

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Our next tool is something new but very helpful when we will be solving for the continued fraction expansions associated with irrational numbers of the form $\sqrt{a}$.

Definition 7

Consider numbers of the form $a + b\sqrt{c}$ where a and b are rational numbers and c is a positive whole number. We define the conjugate of $a + b\sqrt{c}$ as $a - b\sqrt{c}$. We define it as such because, $(a + b\sqrt{c}) \times (a - b\sqrt{c}) = a^2 - b^2c$, where $a^2 - b^2c$ is a rational number.
Note in the same way the conjugate of a number of the form $b\sqrt{c} + a$ is $b\sqrt{c} - a$ and $(b\sqrt{c} + a) \times (b\sqrt{c} - a) = b^2c - a^2$.

Exercise 5.1

What is the conjugate of $(5 - 3\sqrt{2})$? Figure out the product of $(5 - 3\sqrt{2})$ and its conjugate.

Exercise 5.1 Solution

Since $(5 - \sqrt{2})$ contains a minus sign we know that the conjugate will contain a plus sign, so the conjugate of $(5 - 3\sqrt{2})$ is $(5 + 3\sqrt{2})$.

Now we know that $(a+b\sqrt{c})\times(a-b\sqrt{c}) = a^2-b^2c$ by definition 8, so lets start by setting values for a, b, and c. In out case $a = 5$, $b = 3$, and $c = 2$, so $a^2-b^2c = 5^2-3^2\times c = 25-9\times2 = 25-18 = 7$.

Now that we have these two new tools established let's begin doing some examples!

Example 5.1

Identify the irrational number which has the infinite continued fraction expansion [1, 2, 1, 2, 1, 2, ...] = $[1, \bar{2}]$?

Solution: Like in the rational case let's first write out the continued fraction expansion in fraction form, so we get $[1, \bar{2}] = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$. What we will do differently now is that we are going to introduce a variable x to be the irrational number which has the infinite continued fraction expansion $[1, \bar{2}]$. So now we have the following equation to work with:

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$$

Now our goal has become to solve for x.

In the rational case since our continued fraction was finite we could begin manipulating from the denominator upwards, but in this case that is not an option. So instead we use the fact that here our continued fraction is following a pattern which goes on infinitely so we can actually find another instance of x within the expansion itself. In our case we can rewrite our equation as $x = 1 + \cfrac{1}{2 + \frac{1}{x}}$. This is very helpful because now our infinite continued fraction looks like a finite continued fraction, so we can begin solving as we did in the rational case.

Our task is to now simplify the equation $x = 1 + \cfrac{1}{2 + \frac{1}{x}}$. Let's start by simplifying the denominator of the fraction on the right hand side $2 + \frac{1}{x}$. To add two fractions we require a common denominator, in this case the denominator of 2 is 1 and the denominator of $\frac{1}{x}$ is x, so our common denominator is x.

Example 5.1 Continued

Solution Continued: Then we get $2 + \frac{1}{x} = \frac{2x}{x} + \frac{1}{x} = \frac{2x+1}{x}$. So we have shown that $x = 1 + \cfrac{1}{2 + \cfrac{1}{x}} = x = 1 + \cfrac{1}{\cfrac{2x+1}{x}}$. But we know that $\cfrac{1}{\cfrac{2x+1}{x}} = \frac{x}{2x+1}$. So we have now simplified our original problem $x = 1 + \cfrac{1}{2 + \cfrac{1}{x}}$ into the following problem: $x = 1 + \frac{x}{2x+1}$, this is simply now a matter of algebraic manipulations. Let's perform these manipulations one by one to see what is happening:

1. Let's first simplify the right hand side by adding together 1 and $\frac{x}{2x+1}$. Once again to add two fractions we require a common denominator, in this case the denominator of 1 is 1 and the denominator of $\frac{x}{2x+1}$ is $2x+1$, so our common denominator is $2x+1$. So we multiply $1 \times (2x+1) = 2x+1$ so $1 + \frac{x}{2x+1} = \frac{2x+1}{2x+1} + \frac{x}{2x+1} = \frac{2x+1+x}{2x+1} = \frac{3x+1}{2x+1}$. So now we have $x = 1 + \frac{x}{2x+1} = \frac{3x+1}{2x+1}$.
2. Now to solve $x = \frac{3x+1}{2x+1}$ we will multiply both sides by $2x+1$ to cancel out the $2x+1$ in the denominator of the right hand side. This will give us $x \times (2x+1) = 3x+1$.
3. Simplifying $x \times (2x+1)$ we multiply each term in the brackets by x. $x \times 2x = 2x^2$ and $x \times 1 = x$ so $x \times (2x+1) = 2x^2 + x = 3x + 1$.
4. The final manipulation we perform on the equation is to move all the terms from the right hand side to the left hand side! $2x^2 + x = 3x + 1$, we'll first subtract $3x$ from both sides and then subtract 1 from both sides. This gives us $2x^2 + x - 3x - 1 = 3x + 1 - 3x - 1$. The left hand side $2x^2 + x - 3x - 1 = 2x^2 - 2x - 1$ and the right hand side $3x + 1 - 3x - 1 = 0$. So we have $2x^2 - 2x - 1 = 0$

Here is where we want to introduce the quadratic formula, the quadratic we want to plug into the quadratic formula is $2x^2 - 2x - 1 = 0$, so we have $a = 2$, $b = -2$ and $c = -1$.

Example 5.1 Continued

Solution Continued: Plugging this into the quadratic formula gives us

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)} = \frac{2 \pm \sqrt{4+8}}{4} = \frac{2 \pm \sqrt{12}}{4}$$

.

So this tells us that we have two possible solutions for x $\frac{2+\sqrt{12}}{4}$ and $\frac{2-\sqrt{12}}{4}$, but the second possible solution is negative so we know that cannot be equal to $1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$ which is positive.

Therefore the irrational number which has the infinite continued fraction expansion [1, 2, 1, 2, 1, 2, ...] = $[1, \bar{2}]$ is $\frac{2+\sqrt{12}}{4}$.

Activity 3

Now it's your turn to practice!! Get into pairs or groups of three or four and have one person come up and pick an infinite continued fraction expansion. Your task as a group is to identify the irrational number which it represents. Once you're done come and check your answer with me :)!

Example 5.2

Find the infinite continued fraction expansion of $\sqrt{2}$.

Solution:

We are going to start by adding and subtracting one from $\sqrt{2}$ and also multiply $\sqrt{2}$ by one. The reason why we are adding and subtracting one is because the integer part of $\sqrt{2}$ is 1, that is $\sqrt{2} = 1.41421356237$. Notice we are allowed to do this because adding and subtracting one cancel each other out and multiplication by one also changes nothing. So why would we perform this operation- the key is that the why isn't as important as how we do it;

$$\sqrt{2} = 1 + \sqrt{2} - 1 = 1 + (\sqrt{2} - 1) \times 1 = 1 + (\sqrt{2} - 1) \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = 1 + \frac{(\sqrt{2}-1) \times (\sqrt{2}+1)}{\sqrt{2}+1}$$

Now notice what we did is we multiplied by a modified 1 where $1 = \frac{\sqrt{2}+1}{\sqrt{2}+1}$ and we know by Definition 8 that $\sqrt{2}+1$ is the conjugate of $\sqrt{2}-1$. Recall from our definition of the conjugate $(b\sqrt{c} - a) \times (b\sqrt{c} + a) = b^2c - a^2$, we have $b = 1$, $c = 2$ and $a = 1$ so

$$(1\sqrt{2} - 1) \times (1\sqrt{2} = 1) = 1^2 \times 2 - 1^2 = 1 \times 2 - 1 = 2 - 1 = 1$$

Thus $\sqrt{2} = 1 + \frac{(\sqrt{2}-1) \times (\sqrt{2}+1)}{\sqrt{2}+1} = 1 + \frac{1}{1+\sqrt{2}}$.

Similar to when we identified that x was appearing in it's own expression in Example 5.1 here we can replace the $\sqrt{2}$ in the denominator by the expression we have.

Let's more clearly see where this is happening: we have shown so far that:

$\sqrt{2} = 1 + \frac{1}{1+(\sqrt{2})}$, what we want to do is replace the $\sqrt{2}$ in the brackets in the denominator by the entire expression $1 + \frac{1}{1+(\sqrt{2})}$. This gives us the following:

$$\sqrt{2} = 1 + \frac{1}{1 + (1 + \frac{1}{1+(\sqrt{2})})} = 1 + \frac{1}{2 + \frac{1}{1+(\sqrt{2})}}$$

We now notice that because of this replacement we have a new occurrence of $(\sqrt{2})$ in the denominator. We will now once again replace the last occurrence of the $(\sqrt{2})$ in our denominator by the expression we've built so far.

Example 5.2 Continued

Solution Continued:

The new expression we have for $(\sqrt{2})$ is

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + (\sqrt{2})}}$$

. Let's now make the replacement. Once we make this replacement again we get:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + (\sqrt{2})}} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + (1 + \cfrac{1}{2 + \cfrac{1}{1 + (\sqrt{2})}})}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + (\sqrt{2})}}}}$$

.

And here we see our pattern forming- every time we replace the last $\sqrt{2}$ with the expression we built in the step before we add a new string of 2s to the continued fraction expansion. Alas, the infinite continued fraction expansion of $\sqrt{2}$ is $[1, 2, 2, ...] = [1, \bar{2}]$

Activity 4

Now it's your turn to practice!! In your same groups of three or four and have one person come up and pick an irrational number. Your task as a group is to identify the infinite continued fraction expansion which represents it. Once you're done come and check your answer with me :)!

Bonus Section 6: Some Cool Continued Fraction Expansions

We've now gotten comfortable with building and solving for infinite continued fraction expansions we need to understand their importance.

Mathematicians have come to prove that to best approximate an irrational number by a rational number, it is best to do so by using a convergent of its continued fraction expansion! So when we are dealing with irrational numbers like π which are used so commonly used in engineering, physics and many other sciences, we can instead work with the rational number which best approximates it.

Let's talk about a second way of representing an infinite continued fraction! This representation is helpful in the case where the continued fraction does not exhibit a recurring pattern.

Definition 8

Let x be an irrational number and let $x = [a_0, a_1, \cdots]$ be the infinite continued fraction expansion. The n^{th} convergent of x is the rational number $\frac{p_n}{q_n} = [a_0, a_1, \cdots, a_n]$

For example if $x = [1, 2, 3, 4 \cdots]$ then the 3rd convergent of x is $[1, 2, 3, 4]$ since $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, and $a_3 = 4$. What is the 7th convergent of x?

Exercise 6.1

Find the continued fraction expansion of π up to the 6th convergent.

Exercise 6.2

Find the continued fraction expansion of e up to the 6th convergent.

Exercise 6.3

Which irrational number has the infinite continued fraction expansion $[1, 1, 1, \cdots]$?